Math 60 App C.2 (3 days)

Appendix C.2 Systems of Linear Equations in 3 Variables

AKA 3x3 systems

Objectives
1) Recognize a 3x3 system
2) Determine if a given ordered triple is a solution of a given 3x3 system.
3) Solve a 3x3 system
   - primarily by elimination
   - substitution can be used sometimes.

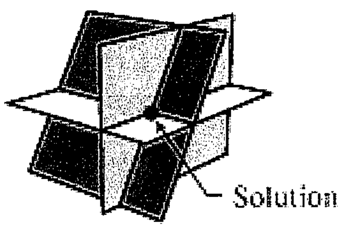
Most of the systems in Appendix C.2 use the variables x, y, and z, though this is not required. They could use other letters, like

Ex:
$$\begin{cases} x + y - 3z = 2 \\ \frac{1}{4}x \qquad - z = -3 \\ -x - 3y + \frac{1}{2}z = 1 \end{cases}$$

$$\begin{cases} a + b - 3c = 2 \\ \frac{1}{4}a \qquad - c = -3 \\ -a - 3b + \frac{1}{2}c = 1 \end{cases}$$

components of a 3x3 system:
- 3 equations connected by a brace $\{$
- 3 different variables, though not every equation need have all 3 variables
- a solution is an ordered triple (x, y, z). which makes all three equations true when substituted.
- when solving algebraically, must use all three equations eventually, though usually we work with two at a time.
- each equation represents a plane in 3-dimensional space.
- an ordered triple represents a single point in 3-dimensional space.

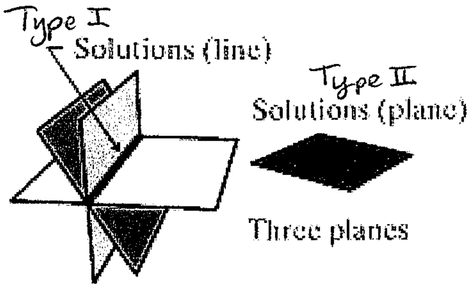
Math 60 Appendix C.2 Systems of Linear Equations in 3 Variables



(a) Consistent system;
one solution

Solve:
an ordered pair

Classify:
consistent
independent



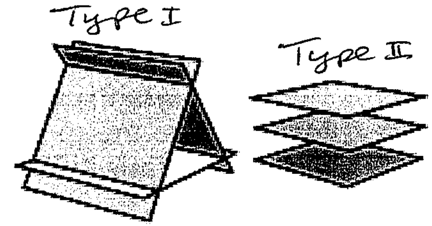
(b) Consistent system;
infinite number of solutions

2 types: In first diagram, the three planes intersect in a line – the line has infinitely many points which are all within all 3 planes.

In second diagram, all three equations are actually the same plane, so every point on one plane is on all three planes. A plane contains infinitely many points.

Solve:
a set with an equation

Classify:
consistent
dependent



(c) Inconsistent system;
no solution

2 types: In the first diagram, any pair of planes intersects, but there are no points that are on all 3 planes.

In the second diagram, the planes are parallel and no plane intersects any other plane

Solve:
"no solution"
or
$\emptyset$

classify:
inconsistent

Math 60 App C.2

① Determine if the ordered triple is a solution of

$$\begin{cases} x + y + z = -2 \\ x + 2y - 3z = 12 \\ 2x - 2y + z = -9 \end{cases}$$

a) $(-1, 2, -3)$

$x = -1$ $y = 2$ $z = -3$

<u>step 1</u>: substitute the values of variables in first eq'n.

$$-1 + 2 + (-3) = -2$$

$$-2 = -2 \checkmark$$

if true continue to next equation.
if false, write "no".

<u>step 2</u>: substitute the values in second eq'n:

$$-1 + 2(2) - 3(-3) = 12$$

$$-1 + 4 + 9 = 12$$

$$12 = 12 \checkmark$$

if true, continue to next equation.
if false, write "no."

<u>step 3</u>: substitute values in third equation.

$$2(-1) - 2(2) + (-3) = -9$$

$$-2 - 4 - 3 = -9$$

$$-9 = -9 \checkmark$$

if true, write yes.
if false, write no.

$\boxed{\text{YES}}$ — This ordered triple satisfies all three equations.

b) $(5, -3, -4)$

<u>step 1</u>: $5 + (-3) + (-4) = -2$

$$5 - 7 = -2 \checkmark$$

<u>step 2</u>: $5 + 2(-3) - 3(-4) = 12$

$$5 - 6 + 12 = 12 \quad \times \quad \boxed{\text{no}}$$

Math 60 App C.2

2) Use elimination method to solve

$$\begin{cases} x + y - z = 6 & (A) \\ 3x - 2y + z = -5 & (B) \\ x + 3y - 2z = 14 & (C) \end{cases}$$

step 1:

GOAL: Eliminate one variable -- the same variable -- twice using two pairs of eq'ns, to get a 2x2 system. There are many options for doing this.

	use A twice: A & B then A & C	use B twice: A & B then B & C	use C twice: B & C then A & C
eliminate x	$A \times (-3) + B$ then $A \times (-1) + C$	$A \times (-3) + B$ then $B + C \times (-3)$	$B + C \times (-3)$; $A \times (-1) + C$
eliminate y	$A \times 2 + B$ then $A \times (-3) + C$	$A \times 2 + B$ then $B \times 3 + C \times 2$	$B \times 3 + C \times 2$; $A \times (-3) + C$
eliminate z	$A + B$ then $A \times (-2) + C$	$\boxed{A + B \text{ then } B \times 2 + C}$	$B \times 2 + C$; $A \times (-2) + C$

When making your choice, keep the following in mind:

1) How will I stay organized?
   - Remember what I am doing
   - Remember which eq'ns I have used
   - Remember which eq'ns I still need to use.

2) How can I keep the arithmetic simple?

For some students, getting lost in the process is a greater worry than arithmetic. For other students this is reversed.

Math 60 App C.2

I choose to eliminate z using eq'n B twice because both sets of arithmetic involve small positive multiples or no multiples at all.

A + B :

$$\begin{array}{r} x + y - z = 6 \\ 3x - 2y + z = -5 \\ \hline 4x - y \quad = 1 \end{array}$$

D ← new eq'n.

B × 2 + C :

$$\begin{array}{r} 6x - 4y + 2z = -10 \\ x + 3y - 2z = 14 \\ \hline 7x - y \quad = 4 \end{array}$$

E ← new eq'n.

Step 2: Solve the resulting 2×2 system

$$\begin{cases} 4x - y = 1 & \text{D} \\ 7x - y = 4 & \text{E} \end{cases}$$

eliminate y by D + E(−1) :

$$\begin{array}{rl} 4x - y = 1 & \text{D} \\ -7x + y = -4 & \text{E} \times (-1) \\ \hline -3x \quad = -3 & \end{array}$$

solve for x:

$$x = 1$$

Subst back for y:

$$4(1) - y = 1 \quad \text{D}$$

$$4 - y = 1$$

$$-y = -3$$

$$y = 3$$

Step 3: Subst x and y values into any of A, B or C to solve for z:

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I choose equation A because the coefficients are smaller numbers and I hope for easy arithmetic:

$$x + y - z = 6$$
$$1 + 3 - z = 6$$
$$4 - z = 6$$
$$-z = 2$$
$$z = -2$$

Step 4: Write as an ordered triple

$\boxed{(1, 3, -2)}$

Note: You may not solve for x, then y, then z, So pay attention to what goes where.

③ Solve by elimination
$$\begin{cases} 2x + y = -4 & \text{(A)} \\ -2y + 4z = 0 & \text{(B)} \\ 3x - 2z = -11 & \text{(C)} \end{cases}$$

When some equations are missing variables, the number of options to consider is reduced, because a 2x2 system can be created using an existing equation.

eliminate x	A × (−3) + C × 2 B
eliminate y	$\boxed{\text{A} \times 2 + \text{B} \quad \text{C}}$
eliminate z	B + C × 2 A

Math 60 App C.2

I choose to eliminate y because the multiplier is small and positive, but it's a different choice from the previous example.

Ⓐ$\times 2$ + Ⓑ:

$$
\begin{aligned}
4x + 2y &= -8 \\
-2y + 4z &= 0 \\
\hline
4x + 4z &= -8
\end{aligned}
$$

This result can be simpler by dividing all terms, both sides of eq'n by 4

$$x + z = -2 \quad Ⓓ$$

A 2x2 system results:

$$
\begin{cases}
x + z = -2 & Ⓓ \\
3x - 2z = -11 & Ⓒ
\end{cases}
$$

Ⓓ $\times 2$ to eliminate z:

$$
\begin{aligned}
2x + 2z &= -4 \qquad Ⓓ\times 2 \\
3x - 2z &= -11 \\
\hline
5x &= -15 \\
x &= -3
\end{aligned}
$$

HINT

If you get a whacked fraction and can't find the error, FINISH ANYWAY to get max partial credit.

Subst into Ⓓ to solve for z:

$$
\begin{aligned}
-3 + z &= -2 \\
z &= 1
\end{aligned}
$$

Subst into an equation containing y to solve for y: either Ⓐ or Ⓑ (because Ⓒ doesn't have y)

$\Rightarrow$ Ⓐ

$$
\begin{aligned}
2(-3) + y &= -4 \\
-6 + y &= -4 \\
y &= 6
\end{aligned}
$$

Ordered triple $(x, y, z) \Rightarrow \boxed{(-3, 6, 1)}$